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Field Equations for Gravity Quadratic in the Curvature

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Vacuum field equations for gravity are studied having their origin in a Lagrangian quadratic in the curvature. The motivation for this choice of the Lagrangian—namely the treating of gravity in a strict analogy to gauge theories of Yang–Mills type—is criticized, especially the implied view of connections as gauge potentials with no dynamical relation to the metric. The correct field equations with respect to variation of the connections and the metric independently are given. We deduce field equations which differ from previous ones by variation of the metric, the torsion, and the nonmetricity from which the connections are built.

1. INTRODUCTION

Fairchild (1976) has proposed a gauge theory of gravity based on earlier elaborations by Stephenson (1958) and Yang (1974). In establishing the Lagrangian of gravity and deriving the field equations, he argues in complete analogy to gauge theories of the Yang–Mills type. So the linear connections $\Gamma_{\mu\nu}{}^{\lambda}$ of affine spaces are viewed as gauge potentials of the gauge group $GL(4, R)$ defining the proper gauge-covariant derivative. The gauge field is given then by the curvature

$$R_{\mu\kappa\lambda}{}^{\varepsilon} = 2\partial_{[\mu}\Gamma_{\kappa]\lambda}{}^{\varepsilon} - 2\Gamma_{[\mu|\lambda}{}^{\rho}\Gamma_{|\kappa]\rho}{}^{\varepsilon} \tag{1.1}$$

To conserve the analogy to gauge theories, the field equations for matter-free gravity are given by variation of the action,

$$I = \int d^4x\, g^{1/2} R_{\mu\kappa\rho}{}^{\sigma} R_{\lambda\varepsilon\sigma}{}^{\rho} g^{\mu\lambda} g^{\kappa\varepsilon} \tag{1.2}$$

with respect to the "gauge potentials" $\Gamma_{\alpha\beta}{}^{\xi}$. Since in Yang–Mills theories the gauge potentials have no dynamical relation to a metric (metric in group

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space), and no symmetries with respect to indices of gauge potentials can be given *a priori*, these facts should also be taken over to the above gauge-analogy gravity. Therefore one gets two distinct sets of field equations: the first by varying (1.2) with respect to the connections $\Gamma_{\alpha\beta}{}^{\xi}$ (with torsion and nonmetricity) viewed as dynamically decoupled from the metric, and the second by varying (1.2) with respect to the metric. There are severe points of objection against this program of Fairchild (1976). Before coming to these, we mention that the equations derived in Fairchild (1976) are wrong. Fairchild has given a hint to this in an erratum to that work. We will derive the correct field equations by varying the above Lagrangian with respect to the connections (with torsion and nonmetricity) independent of the metric, as was tried by Fairchild (1976), and by varying with respect to the metric.

In gauge theories the indices of the gauge potentials $\phi_{\mu i}{}^{j}$ are of different character (marked here by Greek and Latin letters); the Greek index is a space-time index, while the Latin indices are those of the inner symmetry group, the gauge group [e.g., $SU(2)$]. Therefore no assumptions about index symmetries can be made *a priori*. This is in contrast to metric theories of gravity. There the group of space-time transformations and the gauge group are identical and all indices are of the same kind. There are strong physical reasons to split the connections $\Gamma_{\alpha\beta}{}^{\xi}$ into a symmetric and an antisymmetric part with respect to the lower indices. The antisymmetric part $\Gamma_{[\alpha\beta]}{}^{\xi} =: S_{\alpha\beta}{}^{\xi}$ is called torsion and is related to spin in geometric theories of gravity (Bahmann, 1990); it should be omitted if one postulates a strictly local Minkowskian structure of gravity.

In geometric theories of gravity the metric plays the fundamental role of a dynamical variable and not the linear connections, in contrast to gauge theories. All other attempts are more or less mathematical and are inspired by the wish to draw an analogy to gauge theories, but are physically not justified, in contrast to Bahmann (1990). A covariant derivative is defined with the linear connections. The symmetric part of the connections can be separated now into two classes, depending on whether the covariant derivative of the metric itself vanishes or not. The quantity $-\nabla_{\alpha} g_{\beta\gamma} = Q_{\alpha\beta\gamma}$ is called the nonmetricity (∇ means the covariant derivative with respect to the connection). There are strong physical reasons for vanishing nonmetricity (e.g., constant length of 4-vectors during parallel transport).

The most general linear connection $\Gamma_{ij}{}^{k}$ therefore reads (Hehl *et al.*, 1976)

$$\Gamma_{ij}{}^{k} = g^{kl} \Delta_{jil}^{abc} \left(\tfrac{1}{2} \partial_a g_{bc} - g_{cd} S_{ab}{}^{d} + \tfrac{1}{2} Q_{abc} \right) \tag{1.3}$$

with

$$\Delta_{jil}^{abc} := \delta_j^a \delta_i^b \delta_l^c + \delta_i^a \delta_l^b \delta_j^c - \delta_l^a \delta_j^b \delta_i^c \tag{1.4}$$

The first part in (1.3) is the usual Christoffel symbol corresponding to a Riemannian geometry (vanishing torsion and nonmetricity), the second part consists of torsion contributions, and the last part consists of contributions from the nonmetricity. The fundamental dynamical variables are therefore the metric, the torsion, and the nonmetricity *and not the linear connection* built of them. In obtaining field equations for gravity with respect to the action (1.2), these are the quantities after which it should be varied independently.

One cannot argue with the Palatini method, which means that the resulting equations for gravity are the same if one varies independently with respect to the connections and metric making no prior assumption about the connections, or making prior assumptions about the metric dependence of the connections and then varying with respect to the metric alone. Stephenson (1958) has shown that his method is restricted to symmetric connections and the Einstein–Hilbert Lagrangian only.

As a result of all the mentioned differences between gravity and gauge theories, we conclude that varying the proposed action (1.2) with respect to the connections and the metric independently is physically not justified. The analogy of gravity to gauge theories suggested by Fairchild does not exist. Further, the action (1.2) can only serve as a correction to the Einstein–Hilbert action, since the Newtonian limits gets lost (Fairchild, 1976). These corrections may be important in regions of high curvature or in a somewhat quantized version of general relativity. From this point of view and to correct the equations in Fairchild (1976), we calculate the field equations corresponding to the action (1.2).

In Section 2 we give some preliminary fundamental equations and relations needed in the following calculations.

In Section 3 we perform the variations with respect to the connections and metric independently, presenting the corrected equations of Fairchild.

In Section 4 we take into account the dependences of the connections with respect to the metric, torsion, and nonmetricity and present the field equations for the metric, torsion, and nonmetricity.

2. PRELIMINARY RELATIONS

We make the following abbreviations:

$$g := -\det g_{\mu\nu}; \qquad e := \sqrt{g} \tag{2.1}$$

The connections $\Gamma_{ij}{}^{k}$ can be written as

$$\Gamma_{ij}{}^{k} = g^{kl}\Delta_{jil}^{abc}(\tfrac{1}{2}\partial_a g_{bc} - g_{cd}S_{ab}{}^{d} + \tfrac{1}{2}Q_{abc}) \tag{2.2}$$

with

$$\Delta_{jil}^{abc}=\delta_j^a\delta_i^b\delta_l^c+\delta_i^a\delta_l^b\delta_j^c-\delta_l^a\delta_j^b\delta_i^c \tag{2.3}$$

Here

$$S_{ab}{}^{d}:=\Gamma_{[ab]}{}^{d} \tag{2.4}$$

and

$$Q_{abc}:=-\nabla_a g_{bc} \tag{2.5}$$

are the torsion, resp. the nonmetricity, and ∇ means the covariant derivative with respect to the connection (2.2).

Since g is a scalar density of weight -2, we get

$$\nabla_\rho g=g_{,\rho}-2\Gamma_{\rho\xi}{}^{\xi}g \tag{2.6}$$

We make the abbreviation

$$\Gamma_\rho:=\Gamma_{\rho\xi}{}^{\xi} \tag{2.7}$$

(2.6) written for $e=\sqrt{g}$ reads

$$\nabla_\rho e=e_{,\rho}-\Gamma_\rho e \tag{2.8}$$

With (2.2), (2.3), and (2.7) we get

$$\Gamma_\rho=\tfrac{1}{2}g^{\xi l}g_{l\xi,\rho}+Z_\rho \tag{2.9}$$

with

$$Z_\rho:=\tfrac{1}{2}g^{\xi l}Q_{\rho l\xi} \tag{2.10}$$

The variation of g gives

$$\delta g=-g g_{\mu\kappa}\,\delta g^{\mu\kappa} \tag{2.11}$$

Since

$$\delta g^{\mu\kappa}=-g^{\mu\alpha}g^{\kappa\beta}\,\delta g_{\alpha\beta} \tag{2.12}$$

we get

$$\delta g=g g^{\alpha\beta}\,\delta g_{\alpha\beta} \tag{2.13}$$

It follows immediately that

$$\frac{1}{g}g_{,\rho}=g^{\alpha\beta}g_{\alpha\beta,\rho} \tag{2.14}$$

(2.14) in (2.9) gives

$$\Gamma_\rho = \frac{1}{2}\frac{1}{g} g_{,\rho} + Z_\rho \tag{2.15}$$

respectively,

$$\Gamma_\rho = \frac{1}{e} e_{,\rho} + Z_\rho \tag{2.15a}$$

(2.15) in (2.6), resp. (2.15a) in (2.8), give

$$\nabla_\rho g = -2gZ_\rho \tag{2.16}$$

respectively,

$$\nabla_\rho e = -eZ_\rho \tag{2.17}$$

One sees immediately that in the Riemannian case (i.e., vanishing torsion and nonmetricity) $\nabla_\rho g$, resp., $\nabla_\rho e$ vanishes.

Some further useful relations are

$$\Gamma_{\xi\rho}{}^{\xi} = \frac{1}{2}\frac{1}{g} g_{,\rho} + \bar{Z}_\rho \tag{2.18}$$

respectively,

$$\Gamma_{\xi\rho}{}^{\xi} = \frac{1}{e} e_{,\rho} + \bar{Z}_\rho \tag{2.19}$$

with

$$\bar{Z}_\rho = -2S_{\rho\xi}{}^{\xi} + \tfrac{1}{2} g^{\xi l} Q_{\rho\xi l} \tag{2.20}$$

(2.18) with (2.15), resp. (2.19) with (2.15a), give, with the aid of (2.10) and (2.20),

$$\Gamma_\rho - \Gamma_{\xi\rho}{}^{\xi} = 2S_{\rho\xi}{}^{\xi} \tag{2.21}$$

which vanishes of course for vanishing torsion.

3. VARIATION WITH RESPECT TO CONNECTIONS AND METRIC INDEPENDENT OF EACH OTHER

The action I is given by

$$I = \int d^4x \, e\mathscr{L} \tag{3.1}$$

with the Lagrangian density given by

$$\mathscr{L}=R_{\mu\kappa\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho}g^{\mu\lambda}g^{\kappa\varepsilon} \tag{3.2}$$

Here $R_{\mu\kappa\rho}{}^{\sigma}$ means the curvature

$$R_{\mu\kappa\rho}{}^{\sigma}:=2\partial_{[\mu}\Gamma_{\kappa]\rho}{}^{\sigma}+2\Gamma_{[\mu|\alpha}{}^{\sigma}\Gamma_{|\kappa]\rho}{}^{\alpha} \tag{3.3}$$

3.1. Variation with Respect to the Connections

Variation of (3.1) with respect to $\Gamma_{\alpha\beta}{}^{\gamma}$ gives

$$\begin{aligned}\delta I&=\int d^4x\, e\left[\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}\,\delta\Gamma_{\alpha\beta}{}^{\gamma}+\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\,\delta\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}\right]\\ &=\int d^4x\left\{\left[e\left(\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}-\partial_\xi\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\right)\right.\right.\\ &\qquad\left.\left.-e_{,\xi}\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\right]\delta\Gamma_{\alpha\beta}{}^{\gamma}+\left(e\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\,\delta\Gamma_{\alpha\beta}{}^{\gamma}\right)_{,\xi}\right\}\end{aligned}$$

Dropping the 4-divergence leads, with the aid of (2.8), to

$$\begin{aligned}\delta I=\int d^4x&\left[e\left(\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}-\partial_\xi\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}-\Gamma_\xi\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\right)\right.\\ &\left.-(\nabla_\xi e)\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}}\right]\delta\Gamma_{\alpha\beta}{}^{\gamma}\end{aligned} \tag{3.4}$$

With (3.3) we obtain

$$\frac{\partial R_{\mu\kappa\rho}{}^{\sigma}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}R_{\lambda\varepsilon\sigma}{}^{\rho}g^{\mu\lambda}g^{\kappa\varepsilon}=2\Gamma_{\kappa\rho}{}^{\beta}R^{\alpha\kappa}{}_{\gamma}{}^{\delta}+2\Gamma_{\kappa\gamma}{}^{\sigma}R^{\kappa\alpha}{}_{\sigma}{}^{\beta} \tag{3.5}$$

By index transformations $\mu\to\lambda$, $\kappa\to\varepsilon$, $\rho\to\sigma$, $\sigma\to\rho$, $\lambda\to\mu$, and $\varepsilon\to\kappa$ we get

$$\frac{\partial R_{\lambda\varepsilon\sigma}{}^{\rho}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}R_{\mu\kappa\rho}{}^{\sigma}g^{\mu\lambda}g^{\kappa\varepsilon}=2\Gamma_{\varepsilon\sigma}{}^{\beta}R^{\alpha\varepsilon}{}_{\gamma}{}^{\sigma}+2\Gamma_{\varepsilon\gamma}{}^{\rho}R^{\varepsilon\alpha}{}_{\rho}{}^{\beta} \tag{3.6}$$

(3.5) and (3.6) give

$$\frac{\partial\mathscr{L}}{\partial\Gamma_{\alpha\beta}{}^{\gamma}}=4(\Gamma_{\varepsilon\sigma}{}^{\beta}R^{\alpha\varepsilon}{}_{\gamma}{}^{\sigma}+\Gamma_{\varepsilon\gamma}{}^{\rho}R^{\varepsilon\alpha}{}_{\rho}{}^{\beta}) \tag{3.7}$$

Further, we get

$$\frac{\partial R_{\mu\kappa\rho}{}^{\sigma}}{\partial \Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}} R_{\lambda\varepsilon\sigma}{}^{\rho} g^{\mu\lambda} g^{\kappa\varepsilon} = 2R^{\xi\alpha}{}_{\gamma}{}^{\beta} \tag{3.8}$$

Performing the same index transformations as above and adding the result to (3.8) leads to

$$\frac{\partial \mathscr{L}}{\partial \Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}} = 4R^{\xi\alpha}{}_{\gamma}{}^{\beta} \tag{3.9}$$

So we get

$$\partial_{\xi} \frac{\partial \mathscr{L}}{\partial \Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}} = 4R^{\xi\alpha}{}_{\gamma}{}^{\beta}{}_{,\xi} \tag{3.10}$$

We now express the usual partial derivative in (3.10) by the corresponding covariant one and obtain

$$\partial_{\xi} \frac{\partial \mathscr{L}}{\partial \Gamma_{\alpha\beta}{}^{\gamma}{}_{,\xi}} = 4R^{\xi\alpha}{}_{\gamma}{}^{\beta}{}_{;\xi} - 4\Gamma_{\psi\xi}{}^{\psi} R^{\xi\alpha}{}_{\gamma}{}^{\beta} + 4\Gamma_{\varepsilon\sigma}{}^{\beta} R^{\alpha\varepsilon}{}_{\gamma}{}^{\sigma} + 4\Gamma_{\varepsilon\gamma}{}^{\rho} R^{\varepsilon\alpha}{}_{\rho}{}^{\beta} - 4\Gamma_{\xi\psi}{}^{\alpha} R^{\xi\psi}{}_{\gamma}{}^{\beta} \tag{3.11}$$

Inserting (3.11), (3.9), and (3.7) and (3.4), we get

$$\delta I = -4 \int d^4x\, \delta\Gamma_{\alpha\beta}{}^{\gamma} [\nabla_{\xi}(eR^{\xi\alpha}{}_{\gamma}{}^{\beta}) + e(\Gamma_{\xi} - \Gamma_{\psi\xi}{}^{\psi}) R^{\xi\alpha}{}_{\gamma}{}^{\beta} - e\Gamma_{\xi\psi}{}^{\alpha} R^{\xi\psi}{}_{\gamma}{}^{\beta}] \tag{3.12}$$

From the definition of the curvature (3.3) it follows that

$$R^{\xi\alpha}{}_{\gamma}{}^{\beta} = R^{[\xi\alpha]}{}_{\gamma}{}^{\beta} \tag{3.13}$$

From the definition of the connection (2.2) it follows that

$$\Gamma_{ij}{}^{k} = \Gamma_{[ij]}{}^{k} + \Gamma_{(ij)}{}^{k} \tag{3.14}$$

with

$$\Gamma_{[ij]}{}^{k} =: S_{ij}{}^{k}$$

Using (3.14) and (2.21), we obtain for (3.12)

$$\delta I = -4 \int d^4x\, \delta\Gamma_{\alpha\beta}{}^{\gamma} [\nabla_{\xi}(eR^{\xi\alpha}{}_{\gamma}{}^{\beta}) + 2eS_{\xi\psi}{}^{\psi} R^{\xi\alpha}{}_{\gamma}{}^{\beta} - eS_{\xi\psi}{}^{\alpha} R^{\xi\psi}{}_{\gamma}{}^{\beta}] \tag{3.15}$$

and thus for the field equations with respect to the variation of $\Gamma_{\alpha\beta}{}^{\gamma}$

$$\nabla_{\xi}(eR^{\xi\alpha}{}_{\gamma}{}^{\beta})+2eS_{\xi\psi}{}^{\psi}R^{\xi\alpha}{}_{\gamma}{}^{\beta}-eS_{\xi\psi}{}^{\alpha}R^{\xi\psi}{}_{\gamma}{}^{\beta}=0 \tag{3.16}$$

This equation differs from that found by Fairchild (1976), equation (12), in the two terms with explicit torsion. Fairchild's equation therefore is correct only in the case of symmetric connections.

3.2. Variation with Respect to the Metric

Variation of the action (3.1) with respect to the metric (the connections are treated as independent of the metric) gives

$$\delta I=\int d^4x\, R_{\mu\kappa\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho}(\delta g^{\mu\lambda}g^{\kappa\varepsilon}e+g^{\mu\lambda}\delta g^{\kappa\varepsilon}e+\delta e g^{\mu\lambda}g^{\kappa\varepsilon}) \tag{3.17}$$

With (2.11), (2.1), and (2.12) we obtain

$$\delta I=-\int d^4x\, e[R^{\alpha\varepsilon}{}_{\rho}{}^{\sigma}R^{\beta}{}_{\varepsilon\sigma}{}^{\rho}+R^{\lambda\alpha}{}_{\rho}{}^{\sigma}R_{\lambda}{}^{\beta}{}_{\sigma}{}^{\rho}-\tfrac{1}{2}g^{\alpha\beta}R^{\lambda\varepsilon}{}_{\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho}]\delta g_{\alpha\beta} \tag{3.18}$$

Using the antisymmetry property (3.13), we finally get the field equations

$$e[R^{\alpha\varepsilon}{}_{\rho}{}^{\sigma}R^{\beta}{}_{\varepsilon\sigma}{}^{\rho}-\tfrac{1}{4}g^{\alpha\beta}R^{\lambda\varepsilon}{}_{\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho}]=0 \tag{3.19}$$

This equation is identical to that found by Fairchild (1976), equations (25) and (26).[2]

4. TAKING INTO ACCOUNT THE EXPLICIT DEPENDENCE OF THE CONNECTIONS OF METRIC, TORSION, AND NONMETRICITY; FIELD EQUATIONS WITH RESPECT TO THESE QUANTITIES

4.1. Variation with Respect to the Metric

With (3.15) and (3.18) we get

$$\delta I=\int d^4x\,\{-2e[R^{\mu\varepsilon}{}_{\rho}{}^{\sigma}R^{\nu}{}_{\varepsilon\sigma}{}^{\rho}-\tfrac{1}{4}g^{\mu\nu}R^{\lambda\varepsilon}{}_{\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho}]\,\delta g_{\mu\nu}$$
$$-4[\nabla_{\xi}(eR^{\xi\alpha}{}_{\gamma}{}^{\beta})+2eS_{\xi\psi}{}^{\psi}R^{\xi\alpha}{}_{\gamma}{}^{\beta}-eS_{\xi\psi}{}^{\alpha}R^{\xi\psi}{}_{\gamma}{}^{\beta}]\,\delta\Gamma_{\alpha\beta}{}^{\gamma}\} \tag{4.1}$$

[2]Equations (3.16) and (3.19) can also be derived from equations (55) and (56) in Hehl *et al.* (1989) by correspondingly specifying the gravitational Lagrangian, since these equations are general Euler–Lagrange equations.

If we are interested in the field equations with respect to the metric, we have to take the variations of $\Gamma_{\alpha\beta}{}^{\gamma}$ with respect to the metric and its derivatives. With the relation (2.2) we therefore obtain

$$\delta\Gamma_{\alpha\beta}{}^{\gamma}=[-g^{\gamma(\mu|}\Gamma_{\alpha\beta}{}^{|\nu)}-g^{\gamma l}\Delta^{ab(\mu|}_{\beta\alpha l}S_{ab}{}^{|\nu)}]\,\delta g_{\mu\nu} + \tfrac{1}{2}g^{\gamma l}\Delta^{a(bc)}_{\beta\alpha l}\,\delta g_{bc,a} \tag{4.2}$$

Since, μ, ν, resp. b, c, are summation indices and the metric is symmetric, the symmetrization carried out above is implied. As abbreviation we write

$$\nabla_{\xi}(eR^{\xi\alpha}{}_{\gamma}{}^{\beta})+2eS_{\xi\psi}{}^{\psi}R^{\xi\alpha}{}_{\gamma}{}^{\beta}-eS_{\xi\psi}{}^{\alpha}R^{\xi\psi}{}_{\gamma}{}^{\beta}=:U^{\alpha}{}_{\gamma}{}^{\beta} \tag{4.3}$$

We now treat that part in the integral (4.1) which is proportional to $\delta\Gamma_{\alpha\beta}{}^{\gamma}$ and write for this with the abbreviation (4.3)

$$\delta I_{\text{par}}=-4\int d^4x\, U^{\alpha}{}_{\gamma}{}^{\beta}\,\delta\Gamma_{\alpha\beta}{}^{\gamma} \tag{4.4}$$

By insertion of (4.2) in (4.4), the part proportional to the derivative of the metric becomes

$$\begin{aligned}\delta I_{\text{par/par}}&=-4\int d^4x\, U^{\alpha}{}_{\gamma}{}^{\beta}(\tfrac{1}{2}g^{\gamma l}\Delta^{a(bc)}_{\beta\alpha l}\,\delta g_{bc,a})\\ &=-4\int d^4x\,[(U^{\alpha}{}_{\gamma}{}^{\beta}\tfrac{1}{2}g^{\gamma l}\Delta^{a(bc)}_{\beta\alpha l}\,\delta g_{bc})_{,a}\\ &\quad-(U^{\alpha}{}_{\gamma}{}^{\beta}\tfrac{1}{2}g^{\gamma l}\Delta^{a(bc)}_{\beta\alpha l})_{,a}\,\delta g_{bc}]\end{aligned} \tag{4.5}$$

Dropping the 4-divergence and insertion in (4.4) gives

$$\begin{aligned}\delta I_{\text{par}}=-4\int d^4x\,[&-g^{\gamma(\mu|}\Gamma_{\alpha\beta}{}^{|\nu)}U^{\alpha}{}_{\gamma}{}^{\beta}-g^{\gamma l}\Delta^{ab(\mu|}_{\beta\alpha l}S_{ab}{}^{|\nu)}U^{\alpha}{}_{\gamma}{}^{\beta}\\ &-\tfrac{1}{2}(U^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{,a}]\,\delta g_{\mu\nu}\end{aligned} \tag{4.6}$$

Performing in (4.3) the covariant derivative of e using (2.17) gives

$$U^{\alpha}{}_{\gamma}{}^{\beta}=eT^{\alpha}{}_{\gamma}{}^{\beta} \tag{4.7}$$

where the tensor $T^{\alpha}{}_{\gamma}{}^{\beta}$ reads

$$T^{\alpha}{}_{\gamma}{}^{\beta}:=R^{\xi\alpha}{}_{\gamma}{}^{\beta}{}_{;\xi}+2S_{\xi\psi}{}^{\psi}R^{\xi\alpha}{}_{\gamma}{}^{\beta}-S_{\xi\psi}{}^{\alpha}R^{\xi\psi}{}_{\gamma}{}^{\beta}-Z_{\xi}R^{\xi\alpha}{}_{\gamma}{}^{\beta} \tag{4.8}$$

with $Z_{\xi}=\tfrac{1}{2}g^{\sigma\rho}Q_{\xi\rho\sigma}$.

With these abbreviations we get

$$(U^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{,a}=e_{,a}T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l}+e(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{,a} \tag{4.9}$$

The second term on the right side expressed by the covariant derivative gives

$$\begin{aligned}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{,a} &= (T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{;a} \\ &\quad -\Gamma_{a\psi}{}^{a}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{\psi(\mu\nu)}_{\beta\alpha l}) \\ &\quad -\Gamma_{a\psi}{}^{(\mu|}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a\psi|\nu)}_{\beta\alpha l}) \\ &\quad -\Gamma_{a\psi}{}^{(\nu|}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a|\mu)\psi}_{\beta\alpha l})\end{aligned} \tag{4.10}$$

Using (2.19) to express $\Gamma_{\alpha\psi}{}^{\alpha}$ in (4.10) and inserting in (4.9) gives

$$\begin{aligned}(U^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{,a} &= e(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{;a} \\ &\quad - e\bar{Z}_{a}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l}) \\ &\quad - e\Gamma_{a\psi}{}^{(\mu|}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a\psi|\nu)}_{\beta\alpha l}) \\ &\quad - e\Gamma_{a\psi}{}^{(\nu|}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a|\mu)\psi}_{\beta\alpha l})\end{aligned} \tag{4.11}$$

Using the given symmetry in μ, ν in (4.11) and inserting in (4.6) gives

$$\begin{aligned}\delta I_{\text{par}} = -4\int d^4x\, e[&-g^{\gamma(\mu|}\Gamma_{\alpha\beta}{}^{|\nu)}T^{\alpha}{}_{\gamma}{}^{\beta} - g^{\gamma l}\Delta^{ab(\mu|}_{\beta\alpha l}T^{\alpha}{}_{\gamma}{}^{\beta}S_{ab}{}^{|\nu)} \\ &+ \tfrac{1}{2}\bar{Z}_{a}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l}) + \tfrac{1}{2}(S_{ab}{}^{(\nu|}+\Gamma_{(ab)}{}^{(\nu|})T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}(\Delta^{a|\mu)b}_{\beta\alpha l}+\Delta^{ab|\mu)}_{\beta\alpha l}) \\ &- \tfrac{1}{2}(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{;a}]\,\delta g_{\mu\nu}\end{aligned} \tag{4.12}$$

With the aid of the defining relation for Δ^{abc}_{jil} in (2.3) we get

$$\Delta^{a\mu b}_{\beta\alpha l} + \Delta^{ab\mu}_{\beta\alpha l} = 2\delta^{[a|}_{\beta}\delta^{\mu}_{\alpha}\delta^{|b]}_{l} + 2\delta^{\mu}_{\beta}\delta^{[a}_{\alpha}\delta^{b]}_{l} + 2\delta^{(a}_{\beta}\delta^{b)}_{\alpha}\delta^{\mu}_{l} \tag{4.13}$$

We therefore obtain

$$\begin{aligned}&\tfrac{1}{2}(S_{ab}{}^{(\nu|}+\Gamma_{(ab)}{}^{(\nu|})T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}(\Delta^{a|\mu)b}_{\beta\alpha l}+\Delta^{ab|\mu)}_{\beta\alpha l}) \\ &\quad = S_{ab}{}^{(\nu|}(T^{|\mu)}{}_{\gamma}{}^{[a|}g^{\gamma|b]} + T^{[a|}{}_{\gamma}{}^{|\mu)}g^{\gamma|b]}) + \Gamma_{(ab)}{}^{(\nu|}T^{(b|}{}_{\gamma}{}^{|a)}g^{\gamma|\mu)}\end{aligned} \tag{4.14}$$

$$\begin{aligned}&-\Gamma_{\alpha\beta}{}^{(\nu|}T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma|\mu)} - S_{ab}{}^{(\nu|}T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{ab|\mu)}_{\beta\alpha l}) \\ &\quad = -\Gamma_{\alpha\beta}{}^{(\nu|}T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma|\mu)} - S_{ab}{}^{(\nu|}(T^{b}{}_{\gamma}{}^{a}g^{\gamma|\mu)} + T^{a|\mu)}_{\gamma}g^{\gamma b} - T^{|\mu)b}_{\gamma}g^{\gamma a})\end{aligned} \tag{4.15}$$

The addition of (4.14) and (4.15) which is needed in (4.12) vanishes, so that we finally get, with the aid of (2.20),

$$\begin{aligned}\delta I_{\text{par}} = -2\int d^4x\, e[&(-2S_{a\sigma}{}^{\sigma} + \tfrac{1}{2}g^{\sigma\rho}Q_{a\rho\sigma})(T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l}) \\ &- (T^{\alpha}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta\alpha l})_{;a}]\,\delta g_{\mu\nu}\end{aligned} \tag{4.16}$$

(4.16) with (4.4) and (4.10) finally gives the field equations with respect to $g_{\mu\nu}$:

$$\begin{aligned} & R^{\mu\varepsilon}{}_{\rho}{}^{\sigma}R^{\nu}{}_{\varepsilon\sigma}{}^{\rho} - \tfrac{1}{4}g^{\mu\nu}R^{\lambda\varepsilon}{}_{\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho} \\ & + (-2S_{a\sigma}{}^{\sigma} + \tfrac{1}{2}g^{\sigma\rho}Q_{a\rho\sigma})(T^{a}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta a l}) \\ & - (T^{a}{}_{\gamma}{}^{\beta}g^{\gamma l}\Delta^{a(\mu\nu)}_{\beta a l})_{;a} = 0 \end{aligned} \tag{4.17}$$

In the Riemannian case (vanishing torsion and nonmetricity) we get [see the definition of $T^{a}{}_{\gamma}{}^{\beta}$ in (4.8)]

$$R^{\xi(\nu|a|\mu)}{}_{;\xi;a} + \tfrac{1}{2}R^{\mu\varepsilon}{}_{\rho}{}^{\sigma}R^{\nu}{}_{\varepsilon\sigma}{}^{\rho} - \tfrac{1}{8}g^{\mu\nu}R^{\lambda\varepsilon}{}_{\rho}{}^{\sigma}R_{\lambda\varepsilon\sigma}{}^{\rho} = 0 \tag{4.18}$$

4.2. Variation with Respect to the Torsion and Nonmetricity

The starting point is

$$\delta I = -4\int d^4x\, U^{a}{}_{\gamma}{}^{\beta}\,\delta\Gamma_{a\beta}{}^{\gamma} = -4\int d^4x\, eT^{a}{}_{\gamma}{}^{\beta}\,\delta\Gamma_{a\beta}{}^{\gamma} \tag{4.19}$$

With (2.2) the variation of $\Gamma_{ab}{}^{\gamma}$ with respect to the torsion reads

$$\frac{\delta\Gamma_{a\beta}{}^{\gamma}}{\delta S_{\mu\nu}{}^{\sigma}}\,\delta S_{\mu\nu}{}^{\sigma} = -g^{\gamma l}\Delta^{[\mu\nu]c}_{\beta a l}\,g_{c\sigma}\,\delta S_{\mu\nu}{}^{\sigma} \tag{4.20}$$

Inserting (4.20) in (4.19) gives

$$\delta I = +4\int d^4x\, g^{\gamma l}\Delta^{[\mu\nu]c}_{\beta a l}\,g_{c\sigma}\,\delta S_{\mu\nu}{}^{\sigma}\, eT^{a}{}_{\gamma}{}^{\beta} \tag{4.21}$$

and therefore the field equations with respect to the torsion are

$$\Delta^{[\mu\nu]c}_{\beta a l}\,g^{\gamma l}g_{c\sigma}T^{a}{}_{\gamma}{}^{\beta} = 0 \tag{4.22}$$

Following the same procedure with respect to the nonmetricity gives—by taking into account its symmetry—the corresponding field equations for nonmetricity:

$$\Delta^{\mu(\nu\sigma)}_{\beta a l}\,g^{\gamma l}T^{a}{}_{\gamma}{}^{\beta} = 0 \tag{4.23}$$

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